THE CRITICAL CONDITIONS OF THE THERMAL REGIME IN A GENERALIZED COUETTE FLOW

S. A. Bostandzhiyan and A. M. Stolin

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We examine the nonisothermal steady-state flow of a Newtonian fluid between two parallel plates with consideration of the energy dissipation and in the assumption of a hyperbolic relationship between viscosity and temperature under various temperature boundary conditions. It is assumed that the upper plate is moving at a constant speed and that there is a pressure difference across the space between the plates in the direction of plate motion.

The problem of a steady-state Couette flow of a viscous liquid with consideration of the heat of friction in the case of a hyperbolic relationship between the coefficient of viscosity and temperature is examined in a number of references [1-3]. With precisely this relationship between viscosity and temperature Hausenblas investigated the dynamic flow of a liquid in a flat tube [4]. In reference [5], under temperature boundary conditions of the first kind, Regirer examined the flow between two parallel plates, one of which is moving at a constant velocity, while a pressure difference is set up across the clearance in the direction of plate motion. The author did not investigate the derived solution.

It should be noted that in the dynamic flows examined in [4, 5] there is a critical value for the pressure gradient above which it is impossible to have a steady-state regime, a fact not noted by the authors. Kaganov, in [6], was the first to draw attention to the possibility that critical flow regimes could exist. Examining the flow in a flat tube for the general form of the relationship between viscosity and temperature, Kaganov employed the methods of integral analysis to demonstrate that this relationship is hyperbolic or stronger than hyperbolic; there exists a critical value for the pressure gradient above which a steady-state flow regime is impossible.

This paper is devoted to a study of generalized Couette flow, i.e., the flow of a liquid between two parallel plates, one of which is moving at a constant velocity, with a pressure gradient of constant magnitude across the clearance between the plate. It is assumed that the viscosity is a hyperbolic function of temperature. We examine three types of temperature boundary conditions: a) both plates are kept at constant temperatures that, in the general case, are different; b) the upper plate has been thermostated, with Newtonian heat transfer to the ambient medium taking place through the bottom plate; c) Newtonian heat transfer with an identical value for α_1 occurring through both plates.

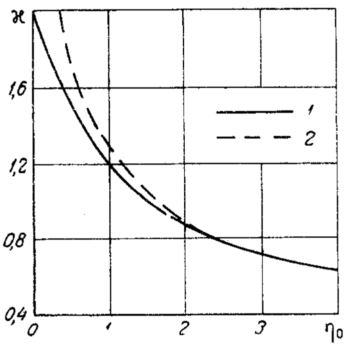

Fig. 1. Limit values of η_0 as a function of $\varkappa$: 1) exact values; 2) approximate values (from (1.16)).

1. Let a layer of a viscous liquid be situated between two infinite plates $y = -h$ and $y = h$, the upper plate moving at a constant velocity v in the direction of the x-axis. A pressure gradient constant in magnitude is found in the clearance between the plate, i.e., $dp/dx = A > 0$. The constant temperatures T_1 and T_2 are specified for the plates. It is assumed that the viscosity is a hyperbolic function of temperature, i.e.,

$$\mu = \frac{\mu_0}{1+\beta^2(T-T_2)} \quad (\mu_0,\ \beta = \text{const}). \tag{1.1}$$

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The system of equations of motion and heat balance with consideration of energy dissipation is written in the form

$$\frac{d}{dy}\left(\mu\frac{dv}{dy}\right)=A,\quad \frac{d^2T}{dy^2}+\frac{\mu}{J\lambda}\left(\frac{dv}{dy}\right)^2=0. \tag{1.2}$$

System (1.2) must be solved for the following boundary conditions:

$$v=0,\ T=T_1 \text{ when } y=-h;\quad v=V,\ T=T_2 \text{ when } y=h. \tag{1.3}$$

From the first equation in (1.2) we have

$$\tau=\mu\frac{dv}{dy}=A(y-y_0). \tag{1.4}$$

The system of equations (1.2) and the boundary conditions (1.3), with consideration of (1.4), can be written in the form

$$\frac{dw}{d\xi}=\xi\theta,\quad \frac{d^2\theta}{d\xi^2}+\varkappa^4\xi^2\theta=0, \tag{1.5}$$

$$\begin{gathered} w=0,\ \theta=\theta_0 \text{ when } \xi=\xi_1=-1-\eta_0; \\ w=\alpha,\ \theta=1 \text{ when } \xi=\xi_2=1-\eta_0. \end{gathered} \tag{1.6}$$

The solution of the second equation in (1.6) is expressed [7] in terms of the Bessel functions

$$\theta=\sqrt{|\xi|}\left[A_1'J_{1/4}\left(\frac{\varkappa^2}{2}\xi^2\right)+B_1'J_{-1/4}\left(\frac{\varkappa^2}{2}\xi^2\right)\right]. \tag{1.7}$$

For the calculations, it is convenient to present [7] the Bessel functions in (1.7) as power series

$$J_p(x)=\frac{\left(\frac{1}{2}x\right)^p}{p!}f_p(x),\quad f_p(x)=\sum_{k=0}^{\infty}\frac{(-1)^k p!}{k!\,(p+k)!}\left(\frac{x}{2}\right)^{2k}. \tag{1.8}$$

Using (1.8), we can write solution (1.7) in the form

$$\theta=A_1\xi f_{1/4}\left(\frac{\varkappa^2}{2}\xi^2\right)+B_1f_{-1/4}\left(\frac{\varkappa^2}{2}\xi^2\right), \tag{1.9}$$

where A_1 and B_1 are new integration constants. Using the boundary conditions (1.6), we obtain

$$\begin{gathered} A_1=\frac{1}{\Delta_1}\left[\theta_0 f_{-1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right)-f_{-1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right)\right], \\ B_1=\frac{1}{\Delta_1}\left[\xi_1 f_{1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right)-\theta_0\xi_2 f_{1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right)\right], \\ \Delta_1=\xi_1 f_{1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right)f_{-1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right)-\xi_2 f_{1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right)f_{-1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right). \end{gathered} \tag{1.10}$$

Having integrated the first equation in (1.5) and using the boundary condition at the bottom plate, we obtain the velocity profile

$$w=F(\xi)-F(\xi_1), \tag{1.11}$$

where

$$\begin{gathered} F(\xi)=A_1g_{1/4}(\xi)+B_1g_{-1/4}(\xi); \\ g_{1/4}(\xi)=\xi^3\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{4}\right)!}{(4k+3)\,k!\left(k+\frac{1}{4}\right)!}\left(\frac{\varkappa^2}{4}\xi^2\right)^{2k}; \\ g_{-1/4}(\xi)=\xi^2\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{4}\right)!}{(4k+2)\,k!\left(k-\frac{1}{4}\right)!}\left(\frac{\varkappa^2}{4}\xi^2\right)^{2k}. \end{gathered} \tag{1.12}$$

Having satisfied the boundary condition at the upper plate at (1.11), we find the equation for the determination of the integration constant:

$$F(\xi_2) - F(\xi_1) = \alpha. \tag{1.13}$$

The temperature and velocity profiles have thus been completely determined.

The Hausenblas results [4] are derived as a special case of the problem under consideration here, when $\alpha = 0$ and $\Theta_0 = 1$.

Let us turn to an investigation of the derived solutions (1.9) and (1.11). We rewrite expression (1.4) for the shearing stress in the form of $\tau/Ah = \eta - \eta_0$. Hence we see that the integration constant η_0 is a quantity which characterizes the shearing stress within the clearance and on the plates. At the same time, η_0 is the ordinate of the point at which the shearing stress vanishes, while the velocity gradient changes sign. If the plates are nonmoving ($\alpha = 0$) and exhibit identical temperature, then $\eta_0 = 0$, while with various temperatures for the plate $\eta_0 \neq 0$ is found within the clearance.

We see from (1.10), (1.12), and (1.13) that $\alpha \to \infty$ as the determinant Δ_1 tends toward zero. With an increase in the plate velocity there will therefore be an increase in η_0, but it will remain smaller than some limit magnitude determined by the first root of the equation $\Delta_1(\varkappa, \eta_0) = 0$. Thus, with an unbounded increase in the plate velocity the shearing stress on the plate does not tend toward ∞ as in the case of isothermal flow, but it tends toward the limit. For pure Couette flow this fact had earlier been established in [1, 2].

It should be noted that with a finite value for α the quantity η_0 determined from (1.13) is a function of the temperature difference Θ_0 at the boundaries of the region, and the limit value of η_0 is independent of Θ_0, since the latter is not included in the expression for the determinant $\Delta_1(\varkappa, \eta_0)$.

The solid line in Fig. 1 shows $\varkappa$ as a function of η_0, determined from the transcendental equation $\Delta_1(\varkappa, \eta_0) = 0$. As the point $(\varkappa, \eta_0)$ approaches the curve, the parameter $\alpha \to -\infty$. It is not difficult to prove that the determinant is an even function of η_0. The point on the curve symmetrical with respect to the $\varkappa$-axis are therefore also roots of the determinant. As the point with the coordinates $(\varkappa, -\eta_0)$ approaches this curve $\alpha \to \infty$. With a reduction in $\varkappa$ the limit toward which the shearing stress tends as $\alpha \to \pm\infty$ increases. As $\varkappa \to 0$, which corresponds to the isothermal flow, this limit is ∞.

Using the asymptotic expressions of the Bessel functions, it is not difficult to derive an analytical expression for the curve that is valid at small values of $\varkappa$. Using (1.9) and the familiar relationships from [7], i.e.,

$$J_{-p}(x) = J_p(x)\cos p\pi - N_p(x)\sin p\pi, \quad (-x)! = \frac{\pi x}{x!\sin \pi x},$$

we can bring the determinant Δ_1 to the form

$$\Delta_1 = \frac{1}{4}\overline{\xi_1\xi_2}\left[J_{1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right) N_{1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right) - J_{1/4}\left(\frac{\varkappa^2}{2}\xi_1^2\right) N_{1/4}\left(\frac{\varkappa^2}{2}\xi_2^2\right)\right]. \tag{1.14}$$

Here N(x) is the Neumann function. We see from Fig. 1 that η_0 increases with a reduction in $\varkappa$, with this change taking place in a manner such that the arguments for the functions in (1.14) increase with a reduction in $\varkappa$. Using the asymptotic expressions for the Bessel and Neumann functions

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

$$N_n(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

we can bring (1.15) to the form

$$\Delta_1 = \frac{1}{\pi\varkappa^2\sqrt{\xi_1\xi_2}}\sin 2\varkappa^2\eta_0. \tag{1.15}$$

Hence we see that the roots of the determinant Δ_1 and, consequently, the limit values for the shearing stress are found from the simple relationship

$$\eta_0 = \frac{\pi}{2\varkappa^2}. \tag{1.16}$$

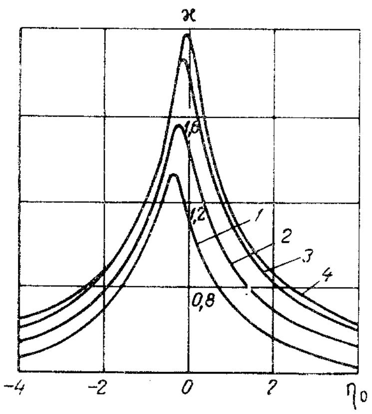

Fig. 2

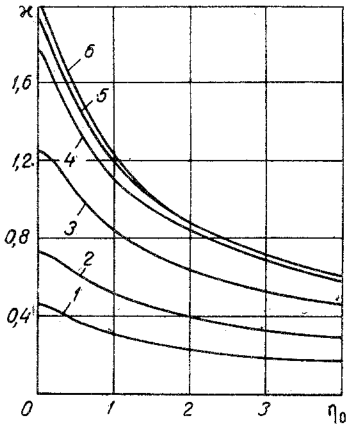

Fig. 3

Fig. 2. Curves for the limit values of η_0 at various values of Bi: 1) Bi = 0; 2) 1; 3) 10; 4) 100.

Fig. 3. Curves for the limit values of η_0 for various values of Bi: 1) Bi = 0.01; 2) 0.1; 3) 1; 4) 10; 5) 100; 6) ∞.

In Fig. 1 function (1.16) is shown by a dashed line. We see from the figure that when $\varkappa = 1$ the curves differ little from each other, while for $\varkappa < 0.8$ they virtually merge.

It is easy to derive a value of $\varkappa$ at which $\eta_0 = 1$, since in this case $\xi_2 = 0$ and the roots of the determinant coincide with the roots of the equation $J_{1/4}(2\varkappa^2) = 0$. The first root – different from zero – of this equation is given by $\varkappa = 1.18$. If $\varkappa < 1.18$, for small values of α the point at which the velocity gradient is equal to zero is situated within the clearance, while for large values of α it is situated outside of the clearance. With $\varkappa \geq 1.18$ it is situated within the clearance for all values of α.

The curve in the case of $\eta_0 = 0$ ends at the axis of ordinate at the point which is the first root of the equation $J_{-1/4}(\varkappa^2/2) = 0$. This value of $\varkappa = \varkappa^* = 2$ is critical. For all $\varkappa < \varkappa^*$ there exists a steady-state flow regime, while for $\varkappa \geq \varkappa^*$ a steady-state regime is impossible, and we have a thermal loss of stability. As $\varkappa$ approaches $\varkappa^*$ there is a pronounced rise in both the temperatures and the velocities. It is interesting to note that the critical value of $\varkappa^*$ is independent both of the temperature difference across the plates and of the plate velocity.

2. Now let the upper plate be thermostated at a temperature T_2, and let the transfer of heat with the ambient medium according to Newton's law proceed through the lower plate. Assuming that the temperature of the ambient medium at considerable distance from the plate is given by T_2, we can write

$$\lambda \frac{dT}{dn} = -\alpha_1 (T - T_2). \tag{2.1}$$

The general solution for the heat-balance equation, as before, will be expressed in the form of (1.9), which must be made subject to the boundary conditions

$$\theta = 1 \text{ when } \eta = 1, \quad \frac{d\theta}{d\eta} = \mathrm{Bi}\,(\theta - 1) \text{ when } \eta = -1. \tag{2.2}$$

With solution (1.9) satisfying boundary conditions (2.2) we find

$$A_2 = \frac{d - b\,\mathrm{Bi}}{\Delta_2}, \quad B_2 = \frac{a\,\mathrm{Bi} - c}{\Delta_2}, \tag{2.3}$$

where

$$\Delta_2 = ad - cb; \quad a = \xi_2 f_{1/4}(\xi_2); \quad b = f_{-1/4}(\xi_1);$$
$$c = \mathrm{Bi}\,\xi_1 f_{1/4}(\xi_1) - \varphi_{1/4}(\xi_1); \quad d = \mathrm{Bi}\, f_{-1/4}(\xi_1) - \varphi_{-1/4}(\xi_1);$$

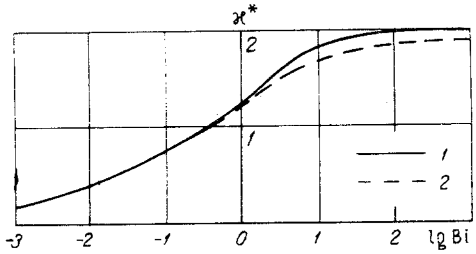

Fig. 4. The values of $\varkappa^*$ as a function of log Bi: 1) exact values; 2) approximate values (from (3.4)).

$$\varphi_{1/4}(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k (4k+1)\left(\frac{1}{4}\right)!}{k!\left(k+\frac{1}{4}\right)!}\left(\frac{\varkappa^2}{4}\xi^2\right)^{2k}; \tag{2.4}$$

$$\varphi_{-1/4}(\xi) = -\frac{\varkappa^4\xi^3}{3}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{3}{4}\right)!}{k!\left(k+\frac{3}{4}\right)!}\left(\frac{\varkappa^2}{4}\xi^2\right)^{2k}.$$

The velocity profile is determined from (1.11), while the integration constant η_0 is obtained from (1.13), provided that the constants A_1 and B_1 in the expression $F(\xi)$ are replaced by A_2 and B_2.

Unlike $\Delta_1(\varkappa, \eta_0)$ the denominator $\Delta_2(\varkappa, \eta_0)$ is not an even function of the argument η_0 and the limit values of η_0 as $\alpha \to +\infty$ and $\alpha \to -\infty$ and fixed $\varkappa < \varkappa^*$ therefore do not coincide. Figure 2 shows the curves giving the limit values of η_0 as functions of the parameter $\varkappa$ for various values of Bi number. All of the curves pass through a maximum which determines the critical value of the parameter $\varkappa$, the left-hand part of the curve determining the limit values of η_0 as $\alpha \to +\infty$, and the right-hand portion of the curve determining the limit value as $\alpha \to -\infty$. When $\varkappa \geq \varkappa^*$ there is no steady-state regime. The case Bi = 0 corresponds to satisfaction at the lower plate of a boundary condition of the second kind, i.e., an absence of heat transfer with the ambient medium. In the case of a thermally insulated bottom plate the critical value is given by $\varkappa^* = 1.35$. With an increase in Bi the curves rise and the apex approaches the axis of ordinates. At the limit Bi $\to \infty$ we have the case of ideal heat transfer through the bottom plate, i.e., the condition of constant temperature is satisfied at the bottom plate.

We see from (2.4) that the roots of the denominator Δ_2 at the limit as Bi $\to \infty$ coincide with the roots of the denominator Δ_1. Even when Bi = 100 the right-hand portion of the curve is virtually coincident with the curve giving the functions relating $\varkappa$ and η_0 in Fig. 1. It follows from the above that at the limit as Bi $\to \infty$, $\varkappa^* = 2$.

Thus, as Bi increases from 0 to ∞ the critical value of $\varkappa^*$ increases from 1.35 to 2. Calculations show that $\varkappa^*$ as a function of Bi can be approximated rather well with a simple fractional-linear function

$$\varkappa^* = \frac{2{,}18 + 2.01\,\mathrm{Bi}}{1.61 + \mathrm{Bi}}. \tag{2.5}$$

The relative error in the determination of $\varkappa^*$ according to (2.5) for all values of Bi does not exceed 0.6%.

3. Let us examine the case in which heat transfer according to Newton's law (2.1) proceeds through both plates to the ambient medium. Assuming that the temperature of the ambient medium at some distance from the plate is given by T_2, we find that the heat-transfer coefficient α_1 is identical for both plates and we have the following temperature boundary condition:

$$\frac{d\theta}{d\eta} = -\mathrm{Bi}(\theta - 1) \text{ when } \eta = 1, \quad \frac{d\theta}{d\eta} = \mathrm{Bi}(\theta - 1) \text{ when } \eta = -1. \tag{3.1}$$

When solution (1.10) satisfies boundary conditions (3.1) we obtain

$$A_3 = \mathrm{Bi}\frac{d - b'}{\Delta_3}, \quad B_3 = \mathrm{Bi}\frac{a' - c}{\Delta_3}, \tag{3.2}$$

where c and d are the same as before, and

$$a' = \mathrm{Bi}\,\xi_2 f_{1/4}(\xi_2) + \varphi_{1/4}(\xi_2),$$
$$b' = \mathrm{Bi}\,f_{-1/4}(\xi_2) + \varphi_{-1/4}(\xi_2), \quad \Delta_3 = a'd - b'c. \tag{3.3}$$

It is not difficult to prove that the determinant $\Delta_3(\varkappa, \eta_0)$ is an even function of the argument η_0; the curves for the limit values of η_0 as $\alpha \to +\infty$ and $\alpha \to -\infty$ are therefore symmetrical with respect to the axis of ordinates.

Figure 3 shows the curves for the limit values of η_0 for various values of Bi as $\alpha \to -\infty$. At the limit, as Bi $\to \infty$, we have the case of ideal heat transfer to the ambient medium, and the curve for Bi = ∞ coincides with the curve in Fig. 1. The maximum of the curves is found on the axis of ordinates and the critical value of the parameter $\varkappa$ is therefore equal to the first root of the equation $\Delta_3(\varkappa, 0) = 0$. The curve showing $\varkappa^*$ as a function of Bi in semilogarithmic coordinates is shown by the solid line in Fig. 4. As Bi diminishes the critical value of the parameter $\varkappa$ does likewise and tends to zero as Bi $\to$ 0. Since at Bi = 0 all the heat evolved remains in the system, the steady-state mode is naturally impossible. It is not difficult to obtain an approximate equation relating $\varkappa^*$ to Bi for small $\varkappa$. If in $\Delta_3(\varkappa, 0)$ we neglect the terms containing $\varkappa$ to a power of over four, we obtain

$$\varkappa^* = \sqrt[4]{\frac{12\,\mathrm{Bi}}{4+\mathrm{Bi}}}. \tag{3.4}$$

The relation between $\varkappa^*$ and Bi defined by Eq. (3.4) is depicted in Fig. 4 (broken line). The relative error in determining $\varkappa^*$ from (3.4) for Bi < 0.4 is less than 1%. With increasing Bi the relative error becomes greater and for Bi = ∞ equals 7.1%.

For the interval $0.4 \leq \mathrm{Bi} < \infty$ we can propose the approximate formula

$$\varkappa^* = 2\sqrt[4]{\frac{\mathrm{Bi}}{5.05+\mathrm{Bi}}}, \tag{3.5}$$

which over the entire indicated interval makes it possible to determine $\varkappa^*$ with a relative error that does not exceed 1%.

In conclusion let us dwell on the applicability of the derived solution. The solution for the problem was derived under two major suppositions: the flow is laminar and the viscosity is a hyperbolic function of temperature. We know that the viscosity of a liquid is described with satisfactory accuracy by this law for a limited range of temperature variations. Calculations show that with an increase in $\varkappa$ the velocities and the maximum temperature initially change slowly, but as they approach the critical value, in the immediate vicinity of that value, we note a pronounced increase in the velocities and in the maximum temperature. As a result, it may turn out that in the vicinity of the critical value of $\varkappa$ the Reynolds number will exceed its critical value and the laminar flow will be disrupted, with the temperature of the liquid in some portion of the clearance between the plates exceeding the limits of applicability for the hyperbolic law.

It is possible to select such values for the clearance between the plates, as well as for the pressure difference and the value of the physical constants of the liquid so that the sharp rise in velocities and temperatures will begin before they attain values destroying the laminar nature of the flow and the hyperbolic function relating viscosity and temperature. A small change in the pressure difference will then correspond to a pronounced variation in temperature and velocity, which may be perceived as a temperature discontinuity in the flow regime. The suitability of the solution for fixed values of $\varkappa$ and α should be judged on the basis of the Reynolds number and on the basis of the maximum heating of the liquid.

NOTATION

v	is the velocity;
T, T_1, and T_2	are the temperatures of the liquid, of the bottom plate, and of the top plate;
V	is the velocity of the top plate;
h	is half the distance between the plates;
$dp/dx = A$	is the pressure gradient;
μ	is the dynamic viscosity of the liquid;
λ	is the coefficient of thermal conductivity for the liquid;
n	is the external normal to the plate;
α_1	is the coefficient of heat transfer between the liquid and the ambient medium;
J	is the mechanical equivalent of heat;
τ	is the shearing stress;
y	is a coordinate;
y_0	is the coordinate of the point at which the shearing stress is equal to zero and it is an integration constant;
$w = \mu_0/Ah^2\, v$	is the dimensionless velocity;

$\Theta = 1 + \beta^2(T - T_2)$ is the dimensionless temperature;
$\eta = y/h$ is the dimensionless coordinate;
$\eta_0 = y_0/h$; $\xi = \eta - \eta_0$; $\alpha = \mu_0 V/Ah^2$; $x^4 = A^2h^4\beta^2/\lambda\mu_0 J$;
$\Theta_0 = 1 + \beta^2(T_1 - T_2)$ are dimensionless parameters;
$\varkappa^*$ is the critical value;
Bi denotes the Biot number.

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